INFLUENCE OF THE WALLS ON OVERHEATING INSTABILITY IN A MAGNETOHYDRODYNAMIC CHANNEL

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 34–39 1965

The mechanism of the overheating instability of magnetohydrodynamic flows is as follows. If the electrical conductivity of the medium depends on the temperature, then a small local increase in temperature may lead, under specific conditions, to an increase in liberation of Joule heat energy, to a further temperature rise, and so to instability.

The overheating instability has been studied before [1, 2] for a uniform unperturbed temperature and without account for the effect of the boundaries of the region. It has been discovered that the growth increment of the disturbances increases as the wavelength increases. However, it is clear that heat conduction through the boundaries of the region occupied by the conducting medium may effect the development of perturbations, primarily of long wavelength perturbations. Below we will examine the simplest problem of the stability of temperature distribution for an electric discharge in a gas between two planes.

We will consider the flow of an incompressible medium with constant velocity $\mathbf{V} = \mathbf{e}_x U$ (it may be taken as zero in the appropriate coordinate system) between two flat electrodes $y = \pm L$, on which constant temperatures and electric potentials are maintained. Let such a temperature distribution be already attained in the flow so that all the Joule heat is conducted away through the walls and the temperature does not vary in the direction of the flow. Such an assumption may be made if the length of the electrodes is much larger that a certain quantity determined by the channel width, the thermal conductivity and the flow velocity. We shall neglect the effect of the magnetic field on the electric current perturbations under consideration, so that the instability will be of a purely "thermal" nature and will not be connected with perturbations of the field by velocity. The instabilities still have this character even in the presence of a uniform magnetic field, when we consider perturbations with a wave vector $\mathbf{k}$ perpendicular to the direction of the field. Actually, in this case

$$\mathrm{rot}\,(\mathbf{j} \times \mathbf{H}) = (\mathbf{H}\nabla)\,\mathbf{j} - \mathbf{H}\,\mathrm{div}\,\mathbf{j} = 0,$$

and the magnetic forces only lead to a redistribution of pressure in the medium.

The temperature of the unperturbed flow T_0 is determined from the equation

$$\varkappa \frac{d^2 T_0}{dy^2} = -\frac{j^2}{\sigma(T_0)},$$

$$T_0(\pm L) = T_w \qquad \left(j = j_y = 2\varphi_e \left(\int_{-L}^{L} \frac{dy}{\sigma} \right)^{-1} \right). \qquad (1)$$

Here $\varkappa$ is the constant thermal conductivity, σ_0 is the electrical conductivity, j is the unperturbed current density, $2\varphi_e$ is the difference of electrode potentials.

One can show, using Maxwell's equations and Ohm's law and making simple estimates, that if the quantity $L^2\sigma/\omega c^2$ (where ω is the characteristic frequency of the problem) is much less than unity, then the perturbations of the electric field $\mathbf{E} = \nabla\varphi$ possess a potential. The equation for φ is obtained from the equation div $\mathbf{j} = 0$ and Ohm's law $\mathbf{j} = \sigma\mathbf{E}$. The second equation of the problem (as regards temperature) comes from the energy equation.

We shall introduce the following dimensionless quantities (the primes will henceforth be omitted):

$$t' = \frac{t}{\rho c_v L^2}, \quad y' = \frac{y}{L}, \quad x' = \frac{x}{L}, \quad f = \varphi \frac{jL}{\varkappa T^*}, \quad T' = \frac{T}{T^*},$$

$$T_0' = \frac{T_0}{T^*}, \quad \alpha^2 = \frac{j^2 L^2}{\varkappa T^* \sigma^*}, \quad \sigma' = \frac{\sigma}{\sigma^*}, \quad \alpha_0{}^2 = \frac{4\varphi_e{}^2 \sigma^*}{\varkappa T^*}.$$

Here $\sigma^* = \sigma(T^*)$ is some characteristic value of the conductivity, and T^* the corresponding temperature. Equation (1) and the linearized equations for the fluctuations of temperature T and potential φ then assume the form

$$\frac{d^2 T_0}{dy^2} = -\frac{\alpha^2}{\sigma}, \quad T_0(\pm 1) = T_w, \quad \alpha^2 = \alpha_0{}^2 \left(\int_{-1}^{1} \frac{dy}{\sigma} \right)^{-2}, \qquad (2)$$

$$\frac{\partial T}{\partial t} = \triangle T + 2\frac{\partial f}{\partial y} + \alpha^2 \frac{T}{\sigma^2}\frac{d\sigma}{dT}, \qquad T(\pm 1) = 0, \qquad (3)$$

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma \frac{\partial f}{\partial y} + \alpha^2 \frac{T}{\sigma}\frac{d\sigma}{dt}\right) = 0, \qquad f(\pm 1) = 0. \qquad (4)$$

For simplicity, in equations (2)–(4) it is assumed that

$$\sigma = \sigma\,[T_0(y)], \qquad d\sigma / dT = (d\sigma / dT)_{T=T_0(y)}.$$

We note that the two final terms of (3) represent the perturbation of Joule heat liberation, and the expression in parentheses in (4) represents the components j_x and j_y of the perturbed current density, respectively.

System (2), which determines the stationary temperature profile in the channel, can always be solved in quadratures. Similar nonlinear problems (but without the integral factor on the right-hand side) arise in the theory of thermal breakdown of dielectrics [3], the theory of thermal explosion [4], and in the investigation of Couette flow with variable viscosity [5]. Employing familiar methods, explained for example in [5], we can show that an inequality of the form $m(T_m) \le \alpha_0^2 \le M(T_m)$ is valid for the parameter α_0^2, where $T_m = T_0(0)$ is the maximal temperature and the form of the functions $m(T_m)$, $M(T_m)$

is determined by the function $\sigma(T)$. In particular, the function $M(T_m) \to 0$ for $T_m \to \infty$, if σ increases sufficiently rapidly as T increases, and consequently a_0^2 is bounded from above and also falls off to zero for $T_m \to \infty$. This also means that the solution of problem (2) exists only for values of a_0^2 less than some critical value. The greatest possible values of a_0^2 for certain functions $\sigma(T)$ are found in [3, 6].

We shall further consider particular solutions of the system (3), (4), having the form

$$T = \theta(y)\, e^{ikx-\lambda t}, \qquad f = \psi(y)\, e^{ikx-\lambda t}.$$

We shall then have for the functions Θ, f,

$$\theta'' + \left(\frac{\alpha^2}{\sigma^2}\frac{d\sigma}{dT} - k^2 + \lambda\right)\theta + 2\psi' = 0, \tag{5}$$

$$(\sigma\psi')' - k^2\sigma\psi + \alpha^2\left(\frac{1}{\sigma}\frac{d\sigma}{dT}\theta\right)' = 0, \tag{6}$$

$$\theta(\pm 1) = 0, \qquad \psi(\pm 1) = 0. \tag{7}$$

In stability investigations we may consider the function $T_0(y)$ and determine the corresponding function $\sigma(T_0)$ from Eq. (2).

Expressing the coefficients in (5), (6) in terms of T_0, we find

$$\theta'' + \left(\lambda - k^2 + \frac{T_0'''}{T_0'}\right)\theta + 2\psi' = 0, \tag{8}$$

$$\left(\frac{\psi'}{T_0''}\right)' + \left(\frac{T_0'''\theta}{T_0'T_0''}\right)' - k^2\frac{\psi}{T_0''} = 0. \tag{9}$$

The problem consists in finding, for given $T_0(y)$ and k, such values of λ for which there exists a nontrivial solution of Eq. (8), (9) satisfying boundary conditions (7). In this connection, the value of λ with the smallest real part is of the greatest interest.

If the system (8), (9) has a solution $\Theta(y)$, $\psi(y)$ for a some λ, then $\Theta(-y)$, $-\psi(-y)$ is also a solution corresponding to the same λ. Thus, when one particular solution corresponds to a given λ

$$\text{either} \quad \theta(-y) = \theta(y), \qquad \psi(-y) = -\psi(y), \tag{10}$$

$$\text{or} \quad \theta(-y) = -\theta(y), \qquad \psi(-y) = \psi(y). \tag{11}$$

If there are several particular solutions, then they may always be chosen so that one of the Eq. (10), (11) is satisfied.

We shall treat the problem under consideration in two limiting cases—for very large and for very small wavelengths.

Small values of k correspond to large wavelengths.* Neglecting terms in (8), (9) containing k^2 and integrating (9) with boundary conditions (7) taken into account, we find

$$\psi = -\frac{T_0'''}{T_0'}\theta + \frac{T_0''}{T_0'(1) - T_0'(-1)}\int_{-1}^{1}\frac{T_0'''}{T_0'}\theta\, dy. \tag{12}$$

The integral on the right-hand of (12) determines the perturbed current density j_y correct to a coefficient.

Keeping in mind Eq. (12), we obtain

$$\theta'' + \left(\lambda - \frac{T_0'''}{T_0'}\right)\theta + \frac{2T_0''}{T_0'(1) - T_0'(-1)}\int_{-1}^{1}\frac{T_0'''}{T_0'}\theta\, dy = 0 \tag{13}$$

from Eq. (8), or

$$\theta'' + \left(\lambda - \frac{\alpha^2}{\sigma^2}\frac{d\sigma}{dT}\right)\theta + \frac{2\alpha^2}{\sigma}\left(\int_{-1}^{1}\frac{1}{\sigma^2}\frac{d\sigma}{dT}\theta\, dy\right)\left(\int_{-1}^{1}\frac{dy}{\sigma}\right)^{-1} = 0,$$

$$\theta(\pm 1) = 0. \tag{14}$$

This boundary value problem will not be Hermitian in the general case.

In accordance with (10), (11) and the remark made above, we need consider only even and odd solutions of Eq. (13). Its last term is zero for odd solutions, so that

$$\theta'' + \left(\lambda - \frac{T_0'''}{T_0'}\right)\theta = 0, \qquad \theta(\pm 1) = 0. \tag{15}$$

This problem is Hermitian and all its eigenvalues λ are real. For $\lambda = 0$ the only odd solution of (15) is $\Theta = T_0'$. In view of Eq. (2) and the inequality $\sigma \geq 0$, we have $T_0'(1) < 0$. Thus, for $\Theta(1)$ to vanish, we must have $\lambda > 0$.

Thus, all odd perturbations are unstable and decay exponentially with time.

It is impossible to carry out an investigation of stability in the general form for even solutions of Eq. (14).

In what follows we shall confine ourselves to two particular cases of conductivity as a function of temperature, when

$$\sigma(T) = Ae^{\beta T} \qquad (A, \beta = \text{const}), \tag{16}$$

or

$$\sigma(T) = \frac{\alpha^2}{\beta^2}\frac{1}{B - T} \qquad (B, \beta = \text{const}). \tag{17}$$

When $\sigma(T)$ is determined by Eq. (16), $T_0'''/T_0' = -\beta T_0''$ and problem (14) becomes Hermitian. All its eigenvalues are real and for $\alpha_0 = 0$ positive. On variation of the parameter α_0 the passage of one of the eigenvalues λ through zero corresponds to the transition to instability.

We shall show that if a simple eigenvalue λ passes through zero for some value of a parameter when the parameter is varied, then for the purposes of solving Eq. (2) this value corresponds to a bifurcation point, and, on the other hand, $\lambda = 0$ corresponds to a bifurcation point. It follows from Eq. (2) that the difference of the two solutions $\Theta = T - T_0$ satisfies the equation

$$\theta'' = -\alpha_0^2\left\{\frac{1}{\sigma(T_0+\theta)}\left[\int_{-1}^{1}\frac{dy}{\sigma(T_0+\theta)}\right]^{-2} - \frac{1}{\sigma(\tau_0)}\left[\int_{-1}^{1}\frac{dy}{\sigma(T_0)}\right]^{-2}\right\} \tag{18}$$

We shall introduce a parameter which charac-

*We can show that in this case the investigation is valid for a compressible medium also.

terizes unambiguously the solutions T_0 which are being investigated for instability. The maximal temperature $T_m = T_0(0)$ may be chosen as such a parameter. If Eq. (18) is further transformed into an integral equation, taking $\Theta''(y)$ as the new unknown, we may make use of theorems contained in [7].

We shall consider the following equation:

$$\mu\theta'' - \frac{\alpha^2}{\sigma^2}\frac{d\sigma}{dT}\,\theta + \frac{2\alpha^2}{\sigma}\left(\int_{-1}^{1}\frac{1}{\sigma^2}\frac{d\sigma}{dT}\,\theta\,dy\right)\left(\int_{-1}^{1}\frac{dy}{\sigma}\right)^{-1} = 0,$$

$$\theta(\pm 1) = 0, \qquad (19)$$

together with Eq. (18).

Here σ and $d\sigma/dT$ depend on $T_0(y)$. For $\mu = 1$ Eq. (19) is the result of linearizing (18) and coincides with (14), if we set $\lambda = 0$ in the latter. If the simple eigenvalue μ of problem (19) passes through unity at the point $T_m = T_m^*$ on variation of T_m, then on the basis of Theorems 2.1, Ch. IV, and 4.7, Ch. II, of [7] we may conclude that $T_m = T_m^*$ is a bifurcation point of the solutions of Eq. (18). The transitions of μ through unity (in problem (19)) and of λ through zero (in problem (14)) come about simultaneously. In order to show this let us suppose that T_m differs little from T_m^*, so that (14) and (19) may be written in the form

$$L\theta = \Delta L\theta - \lambda\theta, \qquad L\theta = \Delta L\theta - (\mu - 1)\,\theta'', \qquad (20)$$

where L is an operator corresponding to $T_m = T_m^*$, ΔL is a "perturbation" of the operator caused by variation of T_m. Equations (20) are solvable if their righthand sides are orthogonal to the eigenfunctions of the operator conjugate with L, and in the given case coincident with L. Thus, for T_m close to T_m^*, we have

$$\lambda \int_{-1}^{1} \theta^2 dy = (1 - \mu) \int_{-1}^{1} \theta'^2 dy. \qquad (21)$$

It follows from Eq. (21) that the transition to instability (passage of λ through zero) is associated with the bifurcation of the solutions of Eq. (2). As investigation of Eq. (2) shows, the bifurcation point of its solutions under condition (16) is the only point [3] of a maximum of the function $a_0(T_m)$ which exists for $\beta > 0$ and is absent for $\beta \le 0$. If $\beta > 0$, then there are two solutions of problem (2) for $\alpha_0 < \alpha_0(T_m^*)$, and for $\alpha_0 > \alpha_0(T_m^*)$ no solution exists, whence it follows [7] that the indices of the respective fixed points of the vector fields corresponding to Eq. (2) in the Banach space of the functions Θ'' have opposite signs, and thus when T_m is varied the quantity μ passes through unity at the point $T = T_m^*$, and λ passes through zero, in accordance with that has gone before. Since the temperature distribution $T_0(y)$ is known to be stable for values of α_0 close to zero, it follows from the explanation just given that it will be stable for $T_m < T_m^*$ and unstable for $T_m > T_m^*$. However, if $\beta < 0$, so that $\alpha_0(T_m)$ is a monotonis function, then the temperature distribution $T_0(y)$ will be stable for any value of T_m. Thus for $\beta < 0$ the solution of Eq. (2) is always stable, and for $\beta > 0$ stable only for $T_m < T_m^*$. For $T_m > T_m^*$ the temperature distribution is unstable with respect to symmetric perturbations of large wave length.

We shall now treat the same problem for a hyperbolic law of conductivity variation (17). It follows from Eq. (2) that

$$\frac{T_0'''}{T_0'} \equiv \beta^2, \qquad T_0 = B - \frac{B - T_w}{\operatorname{ch}\beta}\operatorname{ch}\beta y. \qquad (22)$$

Setting these expressions in (13), we obtain

$$\theta'' + (\lambda - \beta^2)\,\theta + J\,\frac{\beta^3 \operatorname{ch}\beta y}{\operatorname{sh}\beta} = 0,$$

$$\theta(\pm 1) = 0 \quad \left(J = \int_{-1}^{1} \theta\,dy = 1\right). \qquad (23)$$

Since only those solutions of (23) which are even with respect to y are of interest, and, moreover, only those in which the integral in the third term is nonzero, we may normalize Θ making the additional requirement that $J = 1$.

The solution of problem (23) which satisfies the boundary conditions has the form

$$\theta = \frac{\beta^3 \operatorname{cth}\beta}{a^2 + \beta^2}\left(\frac{\cos ay}{\cos a} - \frac{\operatorname{ch}\beta y}{\operatorname{ch}\beta}\right) \qquad (a^2 = \lambda - \beta^2). \qquad (24)$$

The condition $J = 1$ for a or λ gives the equation

$$\frac{2\beta^3 \operatorname{cth}\beta}{a^2 + \beta^2}\left(\frac{\operatorname{tg} a}{a} - \frac{\operatorname{th}\beta}{\beta}\right) = 1. \qquad (25)$$

Whence, after simple transformation,

$$\frac{\operatorname{tg} a}{a} = \frac{3\beta^2 + a^2}{2\beta^3}\operatorname{th}\beta. \qquad (26)$$

The latter equation has extraneous roots $a = \pm i\beta$, since the transition from (25) and (26) involves multiplication by $a^2 + \beta^2$.

For $\beta \to 0$ all roots a_n of Eq. (26) which do not tend to zero are real and lie in the neighborhood of the numbers $a_{n0} = \pi(n - 1/2)$ $(n = 0, +1, \pm 2, \ldots)$, making tg a infinite. Expanding the left-hand side of (26) in a series in a, it is not difficult to verify that this equation does not have roots other than $a = \pm i\beta$ tending to zero for $\beta \to 0$. We shall now trace how the roots of (26) vary with variation of β. As a result of the continuous dependence of (26) on β no new root can arise at a finite point of the complex plane a when β is varied. It is also easy to see that no new roots can come from infinity. Thus it suffices to investigate the behavior of the roots a_n as β increases from zero. The roots a_n are real for $n > 1$ and lie in the intervals $[a_{n0} - \pi/2,\ a_{n0}]$, and the values of λ corresponding to them are positive. The root a_1 belongs to the intercept $[0,\ a_{10}]$ only for $\beta < \beta^*$, where β^* is the root of the equation $3\operatorname{th}\beta = 2\beta$. The root a_1 becomes zero for $\beta = \beta^*$, and for

$B > \beta^*$ a pair of imaginary roots[1] appear $a = \pm i\psi_1$. If we set $a = i\chi$ and write Eq. (26) in the form $F(\chi, \beta) = 0$, then $F(\beta, \beta) = 0$, $F_\chi'(\beta, \beta) > 0$, $F_{\chi\chi}''(\beta, \beta) > 0$.

Consequently, the root of the equation $F(\chi, \beta) = 0$ close to β and specified approximately by the formula

$$\chi_1 \approx \beta + \varepsilon = \beta - \frac{2F_\chi'(\beta, \beta)}{F_{\chi\chi}''(\beta, \beta)} \tag{27}$$

is always less than β. Since it is zero for $\beta = \beta^*$, it is clear that β for any $\beta > \beta^*$ and $\chi_1 \to \beta$ as β increases. For $\beta \gg 1$ we find $\chi_1 \approx \beta - 2\beta^2/3 \operatorname{ch} \beta$, making an approximate calculation of the right hand side of (27).

Positive values of λ also correspond to the imaginary roots $\pm i\chi_1$

$$\lambda = \beta^2 + a^2 = \beta^2 - \chi_1^2 \approx 4\beta^3 / 3 \operatorname{ch}^2 \beta. \tag{28}$$

This proves that for the hyperbolic function (17) the system is stable with respect to very long-wavelength perturbations.

The results we have obtained, and the analogies with other physical phenomena quoted above, allow us to suppose that instability with respect to long-wavelength perturbations in the system under consideration appears in the case when the conductivity increases with an increase in temperature so fast that $\alpha_0(T_m) \to 0$ for $T_m \to \infty$, i.e., when the solution of the stationary problem is nonunique for $\alpha_0 < \alpha^*$ and does not exist for $\alpha_0 > \alpha^*$.

An instability analysis in the case of short-wavelength perturbations is considerably more complex. A relatively simple investigation may be made only in the case when k is sufficiently large, so that we may neglect the first term in Eq. (9) in comparison with the last. If (17) and (22) hold, then we have

$$\left(1 + 2\frac{\beta^2}{k^2}\right)\theta'' - 2\frac{\beta^2}{k^2}\theta' \operatorname{tg} \beta y + \left(\lambda - k^2 + \beta^2 - \frac{2\beta^4}{k^2 \operatorname{ch}^2 \beta y}\right)\theta = 0,$$

$$\theta(\pm 1) = 0,$$

eliminating ψ from (8) and (9).

[1]It must be kept in mind that for real a both parts of (27) are even with respect to a, and so apart from the root a_1 there also exists a root $a_0 = -a_1$ which also vanishes for $\beta = \beta^*$. It is clear that $\lambda > 0$ correspond to the remaining negative roots.

Introducing the new variable $\vartheta = \Theta (\operatorname{ch}\beta y)^{2+k^2/\beta^2}$, we obtain

$$\vartheta'' + \left(\chi - \frac{\delta}{\operatorname{ch}^2 \beta y}\right)\vartheta = 0,$$

$$\vartheta(\pm 1) = 0, \quad \delta = \frac{\beta^4}{k^2}\left(\frac{\beta^2}{k^2} + 1\right)\left(1 + \frac{2\beta^2}{k^2}\right)^{-2},$$

$$\chi = \left[\lambda - k^2 + \beta^2 - \frac{\beta^6}{k^4}\left(1 + 2\frac{\beta^2}{k^2}\right)^{-1}\right]\left(1 + 2\frac{\beta^2}{k^2}\right)^{-1}, \tag{30}$$

from (29).

The solution of Eq. (30) may be represented in terms of elementary functions and so investigated; however, there is no need for this, since for $k \gg 1$, $\beta \sim 1$ we have $\delta \approx \beta^4/k^2 \ll 1$, $\chi \approx \lambda - k^2$. Consequently, the smallest eigenvalue of λ is positive and equal to k^2 in order of magnitude. Thus the temperature distribution in this case is stable not only with respect to long-wavelength perturbations, as was shown above, but also with respect to short-wavelength perturbations with $k \gg 1$.

REFERENCES

1. J. K. Wright, "A temperature instability in magnetohydrodynamic flow," Proc. Phys. Soc., vol. 81, no. 3, pp. 498–505, 1963.

2. B. B. Kadomtsev, "Hydromagnetic stability of a plasma" Problems of Plasma Theory [in Russian], Gosatomizdat, no. 2, pp. 132–176, 1963.

3. W. Franz, Breakdown of Dielectrics [Russian translation], Izd. inostran. lit. 1961.

4. D. A. Frank-Kamenetskii, Diffusion and Heat Transfer in Chemical Kinetics [in Russian], Izd. AN SSSR, 1947.

5. D. D. Joseph "Variable viscosity effects on the flow and stability of flow in channels and pipes," Phys. Fluids, vol. 7, no. 11, pt. 1, pp. 1761–1771, 1964.

6. E. A. Witalis, "Voltage limit of a thermally ionized one-dimensional gas discharge," J. Appl. Phys., vol. 35, no. 12, p. 3617, 1964.

7. M. A. Krasnosel'skii, Topological Methods in the Theory of Non linear Integral Equations [in Russian], Gostekhizdat, 1956.

25 June 1965 Moscow